F. anal. 24.3.2025

- Slides, info (travelling this week, no lec/qu time Thur/Fri)

A. Fourier transf. in $C_m(\mathbb{R}^d)$ (L^1)

$$F[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \cdot e^{-2\pi i x \cdot \xi}\, dx\ , \quad \xi \in \mathbb{R}^d$$

from p.2, 3

~~Radial f'ns:~~ / Gaussians from p.3

Convolutions in $\mathbb{R}^d$:

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

Well-def. for $f, g \in C_m(\mathbb{R}^d)$ (or L^1), and

$$f*g = g*f\ , \quad \|f*g\|_\infty \leq \|f\|_\infty \|g\|_1\ , \quad \|f*g\|_1 \leq \|f\|_1 \cdot \|g\|_1$$

Pf's and other properties as in $\mathbb{R}^1$, including:

Prop. 2: $f, g \in C_m(\mathbb{R}^d)$ (or L^1) –

$\Rightarrow$ $f*g \in C_m(L^1)$ and $\widehat{f*g} = \hat{f} \cdot \hat{g}$ ~~(and f~~

Good kernels in $\mathbb{R}^d$: $\{G_\delta\}_{\delta>0} \subset C_m(\mathbb{R}^d)$ (or L^1) s.t.

(i) $\int_{\mathbb{R}^d} G_\delta = 1$

(ii) $\int_{\mathbb{R}^d} |G_\delta(x)|\, dx \leq M$

(iii) $\int_{|x|>\eta} |G_\delta(x)|\, dx \xrightarrow[\delta \to 0]{} 0$ for $\forall\ \eta > 0$

Prop. 3: (a) If $\int_{\mathbb{R}^d} |f| < \infty$ and f cont. at x,

then $(G_\delta * f)(x) \longrightarrow f(x)$ as $\delta \to 0$.

(b) If $f \in C_m(\mathbb{R}^d)$, then $G_\delta * f \underset{\delta \to 0}{\to} f$ uniformly

[Pf as in $\mathbb{R}^1$].

Gaussians: $K_\delta(x) = \delta^{-\frac{d}{2}} K(\delta^{-\frac{1}{2}} x)$, $K(x) = e^{-\pi|x|^2}$

$K_\delta \in C_m(\mathbb{R}^d)$, radial, realvalued.

Thm. 1: a) $\hat{K}(\xi) = K(\xi) = e^{-\pi|\xi|^2}$

b) $\hat{K}_\delta(\xi) = e^{-\pi\delta|\xi|^2}$

~~c) $\{K_\delta\}_{\delta>0}$ is a good kernel~~

Pf.:

a) $$\hat{K}(\xi) = \int_{\mathbb{R}^d} e^{-\pi x_1^2} \cdots e^{-\pi x_d^2} \cdot e^{-2\pi i x_1 \xi_1} \cdots e^{-2\pi i x_d \xi_d} \, dx_1 \cdots dx_d$$

$$\overset{\text{chng. order int.}}{=} \left(\int_{\mathbb{R}} e^{-\pi x_1^2} e^{-2\pi i x_1 \xi_1} dx_1\right) \cdots \left(\int_{\mathbb{R}} e^{-\pi x_d^2} e^{-2\pi i x_d \xi_d} dx_d\right)$$

$$\overset{\mathbb{R}^1\text{-result}}{=} e^{-\pi \xi_1^2} \cdots e^{-\pi \xi_d^2} = e^{-\pi|\xi|^2}$$

b) From a) and scaling-prop. of $\mathcal{F}_{\mathbb{R}^d}$ □

Thm. 2: $\{K_\delta\}_{\delta>0}$ good kernels

and $K_\delta * f \to f$ unif. for $f \in C_m(\mathbb{R}^d)$

[Pf. as in $\mathbb{R}$, use Prop. 3]

Obs: $f \in C_m \not\Rightarrow \hat{f} \in C_m$ so $\mathcal{F}^{-1}[\hat{f}]$ not def. in general.

B: Fouriertransform in $S(\mathbb{R}^d)$

Schwarz space

$$S(\mathbb{R}^d) := \Big\{ f \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^\alpha| \cdot |\partial_x^\beta f(x)| < \infty \text{ for all } \underbrace{\alpha, \beta \in (\mathbb{N} \cup \{0\})^n}_{\text{multi-indices}} \Big\}$$

$[\, x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \,, \ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \,]$

4.)

Thm. 3: $f, g \in S(\mathbb{R}^d)$

(a) $\mathcal{F}[\partial_x^\alpha f](\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$

(b) $\mathcal{F}[(-2\pi i x)^\alpha f(x)](\xi) = \partial_\xi^\alpha \hat{f}(\xi)$

(c) $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$ (inversion)

(d) $\|\hat{f}\|_2 = \|f\|_2$, $(f, g)_2 = (\hat{f}, \hat{g})_2$ (Plancherel)

Cor. 1:

(a) $f \in S(\mathbb{R}^d) \Rightarrow \hat{f} \in S(\mathbb{R}^d)$ [by Thm 3 a, b]

$$[\text{b} \; \|(2\pi i \xi)^\alpha \partial_\xi^\beta \hat{f}(\xi)\|_\infty = \|\mathcal{F}[\partial_x^\alpha((-2\pi i x)^\beta f(x))](\xi)\|_\infty$$

$$\leq \|\partial_x^\alpha((-2\pi i x)^\beta f(x))\|_1 \overset{f \in S}{\leq} \underbrace{\|(1+|x|^{d+1})\partial_x^\alpha((-2\pi i x)^\beta f(x))\|_\infty}_{<\infty,\ f \in S(\mathbb{R}^d)} \underbrace{\left\| \frac{1}{1+|x|^{d+1}} \right\|_1}_{<\infty} < \infty$$

(b) $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ inv., and

$$\mathcal{F}^{-1}[g(y)](x) = \overline{\mathcal{F}[g](x)} = \mathcal{F}[g](-x) = \mathcal{F}[g(-y)](x)$$

[(a) and Thm. 3 (c)]

(c) $\mathcal{F}$ isometry on $S(\mathbb{R}^d)$ [Thm. 3 (d)]

Pf. of Thm. 3: As in $\mathbb{R}^1$ D

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(a), (b): As in $\mathbb{R}^1$

(c): $\int_{\mathbb{R}^d} f \hat{g} = \int_{\mathbb{R}^d} \hat{f} g$ (change order of integration)

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$$\Rightarrow \int_{\mathbb{R}^d} f \cdot \widehat{\check{K}_\delta} = \int_{\mathbb{R}^d} \hat{f} \cdot \check{K}_\delta$$

$$\widehat{\check{K}_\delta} \overset{\text{Thm. 1}}{=} K_\delta, \quad \check{K}_\delta \overset{\text{Thm 1}}{=} \hat{K}_\delta = e^{-2\pi\delta|\xi|^2}$$

$$\Rightarrow \int_{\mathbb{R}^d} f \cdot K_\delta = \int_{\mathbb{R}^d} \hat{f} \cdot \underbrace{e^{-2\pi\delta|\xi|^2}}_{\xrightarrow[\delta\to 0]{} 1}$$

$\downarrow$ K_δ good kernel $\qquad \downarrow$

$$(*) \qquad f(0) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, d\xi \; (= \mathcal{F}^{-1}[\hat{f}](0))$$

Let $g(y) = f(x+y) \in S(\mathbb{R}^d)$, then

$$f(x) = g(0) = \int_{\mathbb{R}^d} \hat{g} = \int_{\mathbb{R}^d} \mathcal{F}[f(x+y)](\xi)\, d\xi$$

$$= \int_{\mathbb{R}^d} \hat{f}(\xi) \cdot e^{+2\pi i x\cdot\xi}\, d\xi \quad (= \mathcal{F}^{-1}[\hat{f}](x))$$

(d): Pf. as in $\mathbb{R}^1$ $\square$

C. F-transform in $L^2(\mathbb{R}^d)$

<u>Lem. 1:</u> $f \in L^2(\mathbb{R}^d) \Rightarrow \exists \{f_n\}_n \subset S(\mathbb{R}^d)$ s.t. $\|f - f_n\|_2 \to 0$

As in $\mathbb{R}^1$:

$[\quad \hat{f}(\xi) := L^2\text{-}\lim \hat{f}_n(\xi) \qquad (\|\hat{f} - \hat{f}_n\|_2 \to 0)$

is well-def, $\hat{f} \in L^2$, and inherits properties from $S(\mathbb{R}^d)$, like Prop. 1, 2, Thm 3 [more assumptions needed in a), b)]

<u>Rem. 1:</u> All identities now hold in L^2 and a.e.

D. Applications

a) <u>Heat eq'n in $\mathbb{R}^d$:</u>

(1) $\quad u_t = \Delta u = u_{x_1x_1} + \dots + u_{x_dx_d}, \qquad u(t=0) = f(x) \in C_m(\mathbb{R}^d)$

Heat kernel:

$$\mathcal{H}_t^{(d)}(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \quad \text{(heat kernel)}$$

$$u(x,t) = \mathcal{H}_t^{(d)} * f(x)$$

Exercise 12: (i) $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$

(ii) u solves (1)

(iii) $u(x,t) \underset{t\to 0}{\longrightarrow} f(x)$ unif.

b) Heissenberg in $\mathbb{R}^d$:

$$\psi \in S(\mathbb{R}^d),\ \int |\psi|^2 = 1 \Rightarrow \|x\psi(x)\|_2 \cdot \|\xi\hat{\psi}(\xi)\|_2 \geq \frac{d}{4\pi}$$

[Exercise 12]

c) Wave eq'n in $\mathbb{R}^d$:

(2) $u_{tt} = c^2 \Delta u$ in $\mathbb{R}^d \times (0,\infty)$

(3) $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

$\mathcal{F}[(2)]$ in x:

$$\hat{u}_{tt} \overset{(2)}{=} c^2 \left((2\pi i \xi_1)^2 + \dots + (2\pi i \xi_d)^2\right)\hat{u}$$

$$= -4\pi^2 c^2 |\xi|^2 \hat{u}$$

$\Downarrow$ solve ODE in t (chk)

(4) $\hat{u}(\xi,t) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t)$

$\mathcal{F}[(3)]$ in x:

(5) $\hat{u}(\xi,0) = \hat{f}$, $\quad \underset{-A\, 2\pi|\xi|\sin(\,) + B\, 2\pi|\xi|\cos(\,)}{\hat{u}_t(\xi,0)} = \hat{g}$

Solving (4), (5) for A, B:

$$A(\xi) = \hat{f}(\xi) \quad , \quad \hat{B}(\xi) = \frac{\hat{g}}{2\pi|\xi|}$$

Def.:

$$u(x,t) := \mathcal{F}^{-1}[\hat{u}] = \int_{\mathbb{R}^d} \underbrace{\left[\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right]}_{\in S(\mathbb{R}^d) \text{ since } \cos x,\ \frac{\sin x}{x} \in C_b^\infty} e^{2\pi i x\cdot\xi}\, d\xi$$

[Taylor series!]

<u>Thm. 4:</u> $f, g \in S(\mathbb{R}^d)$. Then

(i) $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$, $u(t)\cdot \in S(\mathbb{R}^d)$ for $t>0$

(ii) u solves (2)

(iii) $u(t) \xrightarrow[t\to 0]{} f$, $u_t(t) \xrightarrow[t\to 0]{} g$ unif.

<u>"Pf (ii):"</u>

$$\Delta u = \int [-]\, \Delta e^{2\pi i x\cdot\xi}\, d\xi = \int [-] (-4\pi^2|\xi|^2) e^{-}\, d\xi$$

$$u_{tt} = \int \partial_t^2 [-]\, e^{-}\, d\xi = \int (-4\pi^2|\xi|^2)[-]\, e^{-}\, d\xi \qquad \square$$

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Obs: $\Delta \int(\cdots) = \int \Delta(\cdots)$ by dom. conv. + $\Delta(\cdots) \in C_m$ (chk) $\square$

+ loc. unif. conv. of $D_h^2(\cdots) \to \Delta(\cdots)$